# On some specific cases of quantum state in testing its separability

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This paper shows a simple computational scheme for determining whether a particular quantum state in a specific form is separable across two given sets of qubits. That is, given a set of qubits partitioned into two, it answers the question: does the original state have a separable form as a tensor product of some two other states, which are set up of the two given subsets of qubits?

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## 1. Introduction

In [5], a simple test for the separability of quantum states consisting of nonnegative real quantum probability amplitudes was introduced. Based on that, similar considerations regarding complex amplitudes are presented here in this paper.

## 2. Quantum states equivalence

The so-called classical quantum state description is formed by a vertical vector or, in terms of Dirac's notation, a ket. Each element of the ket is a complex number and the squares of its moduli are the probabilities of observing the corresponding basis states, if measured in the standard basis. In short, a quantum state is expressed with a ket $|\boldsymbol{\varphi}\rangle$ as follows:

$$|\boldsymbol{\varphi}\rangle = \sum_{a=0}^{2^n-1} \delta_a |a\rangle_n \tag{1}$$

where each $|a\rangle_n$ is a basis state

$$|a\rangle_n = [0, \dots, 0, 1, 0, \dots, 0]^T \tag{2}$$

and

$$\sum_{a=0}^{2^n-1} |\delta_a|^2 = 1 \tag{3}$$

with 1 in the $(a+1)$th position and 0's in all the other $2^n - 1$ positions.

The alternative description of quantum states uses the notion of a density matrix $\rho$, defined as

$$\rho = |\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}|. \tag{4}$$

Since the relation between a ket $|\boldsymbol{\varphi}\rangle$ and the corresponding bra $\langle\boldsymbol{\varphi}|$ is that one is the conjugate transpose of the other, we have that, for any phase $x \in [0; 2\pi) + 2k\pi$, where $k = \cdots, -1, 0, 1, 2, \ldots$, the state $\rho$, classically expressed as $|\boldsymbol{\varphi}\rangle$ is equivalent to a state $\rho'$ expressed classically as $|\boldsymbol{\varphi}'\rangle = e^{ix}|\boldsymbol{\varphi}\rangle$ as both are described with the same density matrix. Indeed,

$$\begin{aligned} \rho' = |\boldsymbol{\varphi}'\rangle\langle\boldsymbol{\varphi}'| = \left(e^{ix}|\boldsymbol{\varphi}\rangle\right)\left(e^{-ix}\langle\boldsymbol{\varphi}|\right) = \\ = |\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}| = \rho \end{aligned} \tag{5}$$

Thus, without loss of generality, we can choose one of the equivalent forms of (1) for further consideration. Assuming that, in the polar form, the amplitude $\delta_0 = e^{ix_0}|\delta_0|$, we can always pick up the state $e^{-ix_0}|\boldsymbol{\varphi}\rangle$ instead of $|\boldsymbol{\varphi}\rangle$. This way, the phase shift of the amplitude standing by $|\mathbf{0}\rangle_n = |\overbrace{00\ldots0}^{n\text{ times}}\rangle$ is cancelled and the amplitude itself is nonnegative real. Hence, from now on, if necessary, we can assume that $\delta_0$ is a nonnegative real number, and not complex. Let us call this equivalent state **canonical**.

## 3. Quantum state separability

Assume that the set of qubits $\{1, 2, \ldots, n\}$ is split into two – one consisting of $d$ qubits and the other consisting of $n-d$ qubits. Without loss of generality, we can assume that the qubits are split as $\{1,2,\dots,d\}\cup\{d+1,d+2,\dots,n\}$. Indeed, any permutation of qubits is a unitary operation, hence reversible. If so, we can always rearrange them just before our considerations and revert their order afterward. Now, we can define the $d_1$-$d_2$-separability, where $d_1=d$ and $d_2=n-d$, as follows:

**Definition 1**
A quantum state $|\boldsymbol{\phi}\rangle$ of $n=d_1+d_2$ qubits is $d_1$**-**$d_2$**-separable**, if it is a tensor product of two quantum states: $|\boldsymbol{\phi}_\mathrm{A}\rangle$ of $d_1$ and $|\boldsymbol{\phi}_\mathrm{B}\rangle$ of $d_2$ qubits, respectively. That is, there exist $|\boldsymbol{\phi}_\mathrm{A}\rangle$ and $|\boldsymbol{\phi}_\mathrm{B}\rangle$, such that

$$|\boldsymbol{\phi}\rangle=|\boldsymbol{\phi}_\mathrm{A}\rangle\otimes|\boldsymbol{\phi}_\mathrm{B}\rangle \tag{6}$$

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For the sake of simplicity, let us slightly change the form of (1), exchanging each single-indexed complex number $\delta_a$ with a double-indexed length $\gamma_{kl}$ along with a double-indexed angle $\theta_{kl}$ as follows:

$$\gamma_{kl}=|\delta_a|,$$
$$\theta_{kl}=\mathrm{Arg}\{\delta_a\}$$
$$a=k\cdot 2^{d_2}+l, \tag{7}$$

where $k$ iterates over the basis state indices of the first $d_1$ qubits and $l$ – over the basis state indices of the last $d_2$ qubits (thus $a$ iterates over the basis state indices of all $n=d_1+d_2$ qubits).

Then, we can express (1) in a form that will simplify further notations:

$$|\boldsymbol{\varphi}\rangle=\sum_{a=0}^{2^n-1}\delta_a|a\rangle_n=\sum_{k=0}^{2^{d_1}-1}\sum_{l=0}^{2^{d_2}-1}e^{i\theta_{kl}}\gamma_{kl}|k\rangle|l\rangle. \tag{8}$$

Since the number of qubits in each of the two subgroups ($d_1$ and $d_2$, respectively) is known from the context, the form $|k\rangle|l\rangle$ will be an abbreviation for $|k\rangle_{d_1}\otimes|l\rangle_{d_2}$. In general, if not leading to ambiguity, $|x\rangle$ will stand for some $|x\rangle_d$, where $d$ is the number of qubits known from the context and $x$ within $|\cdot\rangle$ means its binary representation – hence $|x\rangle$ is, in fact, a tensor product of $|0\rangle$'s and $|1\rangle$'s. Moreover, $|\mathbf{0}\rangle$ will stand for some $|00\dots0\rangle$, where the number of 0's is known from the context as well. Remember also that, if not losing generality, we will assume $\theta_{00}=\delta_0=0$ in order for $|\boldsymbol{\varphi}\rangle$ to be in a canonical form.

Assume now that $|\boldsymbol{\varphi}\rangle$ is separable, that is, there exist $|\boldsymbol{\varphi}_A\rangle$ and $|\boldsymbol{\varphi}_B\rangle$ such that $|\boldsymbol{\varphi}\rangle=$ $=|\boldsymbol{\varphi}_A\rangle|\boldsymbol{\varphi}_B\rangle$. Let also:

$$|\boldsymbol{\varphi}_A\rangle=\sum_{k=0}^{2^{d_1}-1}e^{i\xi_k}\alpha_k|k\rangle= =\alpha_0|\mathbf{0}\rangle+\sum_{k=1}^{2^{d_1}-1}e^{i\xi_k}\alpha_k|k\rangle \tag{9}$$

$$|\boldsymbol{\varphi}_B\rangle=\sum_{l=0}^{2^{d_2}-1}e^{i\zeta_l}\beta_l|l\rangle= =\beta_0|\mathbf{0}\rangle+\sum_{l=1}^{2^{d_2}-1}e^{i\zeta_l}\beta_l|l\rangle \tag{10}$$

Of course, again we have just put the amplitudes by both $|\mathbf{0}\rangle$'s to be nonnegative real, that is $\xi_0=\zeta_0=0$. Moreover, the following must hold:

$$\sum_{k=0}^{2^{d_1}-1}|\alpha_k|^2=\sum_{l=0}^{2^{d_2}-1}|\beta_l|^2=1. \tag{11}$$

Now, substituting (9) and (10) into (8),

$$|\boldsymbol{\varphi}_A\rangle|\boldsymbol{\varphi}_B\rangle= =\left(\sum_{k=0}^{2^{d_1}-1}e^{i\xi_k}\alpha_k|k\rangle\right)\left(\sum_{l=0}^{2^{d_2}-1}e^{i\zeta_l}\beta_l|l\rangle\right)= =\sum_{k=0}^{2^{d_1}-1}\sum_{l=0}^{2^{d_2}-1}e^{i(\xi_k+\zeta_l)}\alpha_k\beta_l|k\rangle|l\rangle= =\sum_{k=0}^{2^{d_1}-1}\sum_{l=0}^{2^{d_2}-1}e^{i\theta_{kl}}\gamma_{kl}|k\rangle|l\rangle, \tag{12}$$

we receive, if $|\boldsymbol{\varphi}\rangle$ is $d_1$-$d_2$-separable, that

$$\gamma_{kl}=\alpha_k\beta_l, \tag{13}$$

and

$$\theta_{kl}=_{2\pi}\xi_k+\zeta_l, \tag{14}$$

if $\gamma_{kl}\neq0$. (Here, the congruence relation $a=_x b$ means that there exist an integer $y\in\mathbb{Z}$ such that $a=b+xy$). Note that the case $\gamma_{kl}=$ $=0$ makes the considerations complicated, hence as of now, if not stated otherwise, we will assume that $\gamma_{kl}\neq0$, for every $k$, $l$ and will call such quantum states **inner**. We will come back to the case $\gamma_{kl} = 0$ later.

Due to the previous assumptions, the quantum probability amplitude phase shift $\theta_{00}$, standing by $|\mathbf{0}\rangle_n = |\mathbf{0}\rangle_{d_1} \otimes |\mathbf{0}\rangle_{d_2}$, is 0. Thus, the states we consider follow the rule of always choosing the canonical equivalent state. Even if we have chosen noncanonical $|\boldsymbol{\varphi}_A\rangle$ and/or $|\boldsymbol{\varphi}_B\rangle$, we could always turn the resulting quantum state $|\boldsymbol{\varphi}\rangle = |\boldsymbol{\varphi}_A\rangle|\boldsymbol{\varphi}_B\rangle$ to be so, by multiplying it by $e^{-i\theta_{00}} = e^{-i(\xi_0+\zeta_0)}$. It is easy to see that this is equivalent to turning $|\boldsymbol{\varphi}_A\rangle$ into a canonical state by multiplying it by $e^{-i\xi_0}$ and, at the same time, turning $|\boldsymbol{\varphi}_B\rangle$ into a canonical state by multiplying it by $e^{-i\zeta_0}$. Our task to find the separation for a quantum state is thus equivalent to find two canonical states $|\boldsymbol{\varphi}_A\rangle$ and $|\boldsymbol{\varphi}_B\rangle$, the tensor product of whose form a presumably canonical state $|\boldsymbol{\varphi}\rangle$. Indeed, every problem of noncanonical state $|\boldsymbol{\varphi}'\rangle$ can be reformulated as finding the following canonical separation for a canonical state:

$$|\boldsymbol{\varphi}'\rangle = e^{-i\theta_{00}}|\boldsymbol{\varphi}\rangle = e^{-i\theta_{00}}(|\boldsymbol{\varphi}_A\rangle \otimes |\boldsymbol{\varphi}_B\rangle) = \\ = e^{-i\xi_0}|\boldsymbol{\varphi}_A\rangle \otimes e^{-i\zeta_0}|\boldsymbol{\varphi}_B\rangle, \qquad (15)$$

where $|\boldsymbol{\varphi}\rangle$, $|\boldsymbol{\varphi}_B\rangle$, and $|\boldsymbol{\varphi}_B\rangle$ are in their canonical forms.

## 4. Quantum state separation

Assume we have a separable quantum state $|\boldsymbol{\varphi}\rangle = |\boldsymbol{\varphi}_A\rangle|\boldsymbol{\varphi}_B\rangle$. Let also $|\boldsymbol{\varphi}'_A\rangle$ and $|\boldsymbol{\varphi}'_B\rangle$ be defined as follows (compare to [5]):

$$|\boldsymbol{\varphi}'_A\rangle = \sum_{k=0}^{2^{d_1-1}} e^{i\xi'_k}\alpha'_k|k\rangle, \qquad (16)$$

$$|\boldsymbol{\varphi}'_B\rangle = \sum_{l=0}^{2^{d_2-1}} e^{i\zeta'_l}\beta'_l|l\rangle, \qquad (17)$$

where

$$\alpha'_k = \sqrt{\sum_{l=0}^{2^{d_2-1}} \gamma_{kl}^2}, \qquad (18)$$

$$\beta'_l = \sqrt{\sum_{k=0}^{2^{d_1-1}} \gamma_{kl}^2}, \qquad (19)$$

$$\xi'_k = \theta_{k0} \qquad (20)$$

$$\zeta'_l = \theta_{0l}. \qquad (21)$$

It is easy to see that $|\boldsymbol{\varphi}'_A\rangle$ and $|\boldsymbol{\varphi}'_B\rangle$ are well--defined quantum probability states in a canonical form ($\xi'_0 = \zeta'_0 = \theta_{00} = 0$). Indeed, based on the form (1) together with the condition (3), we have

$$\sum_{k=0}^{2^{d_1-1}} \left|e^{i\xi'_k}\alpha'_k\right|^2 = \sum_{k=0}^{2^{d_1-1}} |\alpha'_k|^2 = \\ = \sum_{k=0}^{2^{d_1-1}} \sum_{l=0}^{2^{d_2-1}} \gamma_{kl}^2 = 1, \qquad (22)$$

and similarly

$$\sum_{l=0}^{2^{d_2-1}} \left|e^{i\zeta'_l}\beta'_l\right|^2 = \sum_{l=0}^{2^{d_2-1}} |\beta'_l|^2 = \\ = \sum_{l=0}^{2^{d_2-1}} \sum_{k=0}^{2^{d_1-1}} \gamma_{kl}^2 = \sum_{k=0}^{2^{d_1-1}} \sum_{l=0}^{2^{d_2-1}} \gamma_{kl}^2 = 1. \qquad (23)$$

Hence, the norm of both states $|\boldsymbol{\varphi}'_A\rangle$ and $|\boldsymbol{\varphi}'_B\rangle$ is 1. This shows that these two states hold (3) accordingly. Moreover, for a $d_1$-$d_2$-separable state $|\boldsymbol{\varphi}\rangle = |\boldsymbol{\varphi}_A\rangle|\boldsymbol{\varphi}_B\rangle$,

$$\alpha'_k = \sqrt{\sum_{l=0}^{2^{d_2-1}} \gamma_{kl}^2} = \sqrt{\sum_{l=0}^{2^{d_2-1}} \alpha_k^2\beta_l^2} = \\ = \alpha_k\sqrt{\sum_{l=0}^{2^{d_2-1}} \beta_l^2} = \alpha_k, \qquad (24)$$

and

$$\beta'_l = \sqrt{\sum_{k=0}^{2^{d_1-1}} \gamma_{kl}^2} = \sqrt{\sum_{k=0}^{2^{d_1-1}} \alpha_k^2\beta_l^2} = \\ = \beta_l\sqrt{\sum_{k=0}^{2^{d_1-1}} \alpha_k^2} = \beta_l. \qquad (25)$$

The (24) and (25) show that, for a separable $|\boldsymbol{\varphi}\rangle$, $|\boldsymbol{\varphi}_A\rangle$ and $|\boldsymbol{\varphi}_B\rangle$ coincide with $|\boldsymbol{\varphi}'_A\rangle$ and $|\boldsymbol{\varphi}'_B\rangle$, respectively, in the moduli of their amplitudes.

Now, it remains to show that they coincide also in phase shifts, i.e. equal up to the period

$2\pi$. Indeed, according to (14), (20), and (21) we have

$$\xi_k' = \theta_{k0} =_{2\pi} \xi_k + \zeta_0 = \xi_k \tag{26}$$

and

$$\zeta_l' = \theta_{0l} =_{2\pi} \xi_0 + \zeta_l = \zeta_l \tag{27}$$

This basically gives proof for the following theorem:

**Theorem 1**
An inner quantum state $|\boldsymbol{\varphi}\rangle$ is $d_1$-$d_2$-separable if and only if $|\boldsymbol{\varphi}\rangle = |\boldsymbol{\varphi}_A'\rangle|\boldsymbol{\varphi}_B'\rangle$ (up to their canonical forms).

**Proof**
It has already been shown that, for a separable $|\boldsymbol{\varphi}\rangle$, $|\boldsymbol{\varphi}_A'\rangle$ equals to $|\boldsymbol{\varphi}_A\rangle$ and $|\boldsymbol{\varphi}_B'\rangle$ equals to $|\boldsymbol{\varphi}_B\rangle$, hence $|\boldsymbol{\varphi}_A'\rangle|\boldsymbol{\varphi}_B'\rangle$ equals to $|\boldsymbol{\varphi}_A\rangle|\boldsymbol{\varphi}_B\rangle = |\boldsymbol{\varphi}\rangle$. On the other hand, if $|\boldsymbol{\varphi}\rangle$ equals to $|\boldsymbol{\varphi}_A'\rangle|\boldsymbol{\varphi}_B'\rangle$, then it is separable ($d_1$-$d_2$-separable) by definition. ■

We can also extend the above theorem to the general notion of separability:

**Theorem 2**
An inner quantum state $|\boldsymbol{\varphi}\rangle$ is (generally) separable if and only if there exists $d_1$ and $d_2$, such that $n = d_1 + d_2$ and $|\boldsymbol{\varphi}\rangle_n = |\boldsymbol{\varphi}_A'\rangle_{d_1} \otimes |\boldsymbol{\varphi}_B'\rangle_{d_2}$ (up to their canonical forms).

**Proof**
This directly results from theorem 1 and the definition of separability (see e.g. [1], [6]). ■

**Example 1**
Let $|\widetilde{\boldsymbol{\varphi}}\rangle = \frac{i}{2}|00\rangle - \frac{i}{2}|11\rangle - \frac{i}{2}|11\rangle + \frac{i}{2}|11\rangle$. We consider $|\boldsymbol{\varphi}\rangle = \frac{1}{i}|\widetilde{\boldsymbol{\varphi}}\rangle = \left[\frac{1}{2};\frac{-1}{2};\frac{-1}{2};\frac{1}{2}\right]^T$. We have
$\alpha_0' = \sqrt{\gamma_{00}^2 + \gamma_{01}^2} = \frac{1}{\sqrt{2}}$, $\xi_0' = \theta_{00} = 0$,
$\alpha_1' = \sqrt{\gamma_{10}^2 + \gamma_{11}^2} = \frac{1}{\sqrt{2}}$, $\xi_1' = \theta_{10} = \pi$,
$\beta_0' = \sqrt{\gamma_{00}^2 + \gamma_{10}^2} = \frac{1}{\sqrt{2}}$, $\zeta_0' = \theta_{00} = 0$,
$\beta_1' = \sqrt{\gamma_{01}^2 + \gamma_{11}^2} = \frac{1}{\sqrt{2}}$, $\zeta_1' = \theta_{01} = \pi$.
Next, $|\boldsymbol{\varphi}_A'\rangle = |\boldsymbol{\varphi}_B'\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ and, finally,
$i|\boldsymbol{\varphi}_A'\rangle|\boldsymbol{\varphi}_B'\rangle = \left[\frac{i}{2};\frac{-i}{2};\frac{-i}{2};\frac{i}{2}\right]^T = |\widetilde{\boldsymbol{\varphi}}\rangle$. We conclude that $|\widetilde{\boldsymbol{\varphi}}\rangle$ is separable: $|\widetilde{\boldsymbol{\varphi}}\rangle = \left(\frac{i}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \otimes \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$. ▲

**Example 2**
Let $|\widetilde{\boldsymbol{\varphi}}\rangle = \frac{i}{2}|00\rangle - \frac{i}{2}|11\rangle + \frac{i}{2}|11\rangle + \frac{i}{2}|11\rangle$. Thus, we consider $|\boldsymbol{\varphi}\rangle = \frac{1}{i}|\widetilde{\boldsymbol{\varphi}}\rangle = \left[\frac{1}{2};\frac{-1}{2};\frac{1}{2};\frac{1}{2}\right]^T$. We have
$\alpha_0' = \sqrt{\gamma_{00}^2 + \gamma_{01}^2} = \frac{1}{\sqrt{2}}$, $\xi_0' = \theta_{00} = 0$,
$\alpha_1' = \sqrt{\gamma_{10}^2 + \gamma_{11}^2} = \frac{1}{\sqrt{2}}$, $\xi_1' = \theta_{10} = 0$,
$\beta_0' = \sqrt{\gamma_{00}^2 + \gamma_{10}^2} = \frac{1}{\sqrt{2}}$, $\zeta_0' = \theta_{00} = 0$,
$\beta_1' = \sqrt{\gamma_{01}^2 + \gamma_{11}^2} = \frac{1}{\sqrt{2}}$, $\zeta_1' = \theta_{01} = \pi$,
yielding
$|\boldsymbol{\varphi}_A'\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$,
$|\boldsymbol{\varphi}_B'\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$,
concluding that $|\widetilde{\boldsymbol{\varphi}}\rangle$ is not separable, albeit very similar to the separable state from example 1, since $|\widetilde{\boldsymbol{\varphi}}\rangle = i|\boldsymbol{\varphi}\rangle \neq i(|\boldsymbol{\varphi}_A'\rangle|\boldsymbol{\varphi}_B'\rangle)$. ▲

## 5. Phase shifts

The formulas (20) and (21) were chosen arbitrarily so that, in view of (26), (27), and (14), taking into account that $\theta_{00} = \xi_0 = \zeta_0 = 0$, and for any $k, k', l, l'$, the following would hold for every separable state:

$$\theta_{kl} - \theta_{k'l} =_{2\pi} \theta_{kl'} - \theta_{k'l'}. \tag{28}$$

The above guarantees that there exist such $(\xi_k)_{k=0}^{2^{d_1}-1}$ and $(\zeta_l)_{l=0}^{2^{d_2}-1}$ that $\theta_{kl} =_{2\pi} \xi_k + \zeta_l$, as per (14). However, (20) and (21) could have been chosen differently. We only require that, for a separable state, $\xi_k' =_{2\pi} \xi_k$ and $\zeta_l' =_{2\pi} \zeta_l$. Thus, we are free to formulate it as

$$\xi_k' = \theta_{kl} - \theta_{0l}, \text{ for any } l \tag{29}$$
$$\zeta_l' = \theta_{0l}. \tag{30}$$

Even more, instead of (29), we can take any weighted average

$$\xi_k' = \sum_{l=0}^{2^{d_2}-1} w_l(\theta_{kl} - \theta_{0l}), \tag{31}$$

where

$$\sum_{l=0}^{2^{d_2}-1} w_l = 1. \tag{32}$$

Note that (32) does not require $w_l$'s to form a convex combination of $(\theta_{kl} - \theta_{0l})$'s. Since, in view of (28), $\theta_{kl} - \theta_{0l} =_{2\pi} \theta_{k0} - \theta_{00}$ (taking $l' = 0$), we derive from (31) and (32)

$$\xi'_k = \sum_{l=0}^{2^{d_2}-1} w_l(\theta_{kl} - \theta_{0l}) =_{2\pi}$$
$$=_{2\pi} (\theta_{k0} - \theta_{00}) \sum_{l=0}^{2^{d_2}-1} w_l =$$
$$= \theta_{k0} - \theta_{00} = \theta_{k0}. \quad (33)$$

This obviously coincides with (20) (up to the period $2\pi$).

## 6. States with zero amplitudes

The formula (14) for $\theta_{kl}$ is correct, if the corresponding amplitude length is not 0, i.e. $\gamma_{kl} \neq 0$. However, this is not always the true. If $\gamma_{kl}$ happens to be 0, the test for separability is not that straightforward and needs some reasoning, sometimes a little bit tricky. Note that (14) allows $\theta_{kl}$'s to be anything we like and the quantum state in question can be separable, if all of these $\theta_{kl}$'s hold (28) – apart from (13) and (14). Thus, let us treat all such $\theta_{kl}$'s, for which $\gamma_{kl} = 0$, as variables and build a system of equations to check if it has any solution. Remember that, if $\kappa$ is a solution for some variable $\theta_{kl}$, then $\kappa \pm 2\pi$ is so as well.

**Example 3**

Let $|\widetilde{\boldsymbol{\varphi}}\rangle = \frac{i}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = i\left[\frac{1}{\sqrt{2}}|00\rangle + -\frac{i}{\sqrt{2}}|11\rangle\right]$, i.e. we consider $|\boldsymbol{\varphi}\rangle = = \left[\frac{1}{\sqrt{2}}; 0; 0; \frac{-i}{\sqrt{2}}\right]^T$. It gives $\gamma_{00} = \gamma_{11} = \frac{1}{\sqrt{2}}$, $\gamma_{01} = = \gamma_{10} = 0$, $\theta_{00} = 0$, $\theta_{11} = -\pi$. Since $\gamma_{01}$ and $\gamma_{10}$ are 0, we need to solve the following equations for $\theta_{01}$ and $\theta_{10}$: $\theta_{01} + \theta_{10} =_{2\pi} \overbrace{\theta_{00} + \theta_{11} = -\pi}^{\text{const}}$. Since this is the only equation in a system, this further reduces to $\theta_{01} + \theta_{10} = \pi$ (as this is enough to find just one sum $\theta_{01} + \theta_{10}$ instead of infinitely many of them, differing by multiples of $2\pi$). Let $\theta_{01}$ be some arbitrarily chosen value $\kappa \in \in [0; 2\pi)$, then $\theta_{10} = \pi - \theta_{01} = \pi - \kappa$. Next, based on (18), (19), (20), and (21) we obtain:

$\alpha'_0 = \sqrt{\gamma_{00}^2 + \gamma_{01}^2} = \frac{1}{\sqrt{2}}$, $\xi'_0 = \theta_{00} = 0$,

$\alpha'_1 = \sqrt{\gamma_{10}^2 + \gamma_{11}^2} = \frac{1}{\sqrt{2}}$, $\xi'_1 = \theta_{10} = \pi - \kappa$,

$\beta'_0 = \sqrt{\gamma_{00}^2 + \gamma_{10}^2} = \frac{1}{\sqrt{2}}$, $\zeta'_0 = \theta_{00} = 0$,

$\beta'_1 = \sqrt{\gamma_{01}^2 + \gamma_{11}^2} = \frac{1}{\sqrt{2}}$, $\zeta'_1 = \theta_{01} = \kappa$.

This gives $|\boldsymbol{\varphi}'_A\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{e^{-i\kappa}}{\sqrt{2}}|1\rangle$, $|\boldsymbol{\varphi}'_B\rangle = = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\kappa}}{\sqrt{2}}|1\rangle$ and $i|\boldsymbol{\varphi}'_A\rangle|\boldsymbol{\varphi}'_B\rangle = = i\left(\frac{1}{2}|00\rangle + +\frac{e^{i\kappa}}{2}|01\rangle - \frac{e^{-i\kappa}}{2}|10\rangle - \frac{1}{2}|11\rangle\right) \neq \neq i|\boldsymbol{\varphi}\rangle = |\widetilde{\boldsymbol{\varphi}}\rangle$ for any $\kappa$. Finally, we conclude that $|\widetilde{\boldsymbol{\varphi}}\rangle$ is not separable. ▲

## 7. Conclusions

The formula (28) imposes a necessary condition on phase shifts of quantum probability amplitudes for separable states. It means that, in order to prove that some quantum state is not separable ($d_1$-$d_2$-separable) it may suffice to show that, for some $k, k', l, l'$, (28) does not hold.

**Example 4**

Let a quantum state be as in example 3. We have $\theta_{00} = \theta_{11} = 0$, $\theta_{10} = \pi - \kappa$, and $\theta_{01} = \kappa$. If the state was separable, formula (28) would hold and $\theta_{10} - \theta_{00} =_{2\pi} \theta_{11} - \theta_{01}$ ($k = 1, k' = = 0, l = 0, l' = 1$), that is $\pi - \kappa =_{2\pi} -\kappa$, which is equivalent to $\pi =_{2\pi} 0$. But this is not the case, which, in this simple manner, shows that the state in question is not separable. ▲

Similarly, independently of the phase shifts, it is required for a separable state that the moduli of its amplitudes hold $\gamma_{kl} = \alpha'_k \beta'_l$, where $\alpha'_k$ and $\beta'_l$ – as per (18) and (19), respectively.

**Example 5**

Let a quantum state be as in example 3. We have $\gamma_{00} = \frac{1}{\sqrt{2}}$, $\gamma_{01} = 0$, $\gamma_{10} = 0$, and $\gamma_{11} = \frac{-i}{\sqrt{2}}$. Next, $\alpha'_0 = \alpha'_1 = \beta'_0 = \beta'_1 = \frac{1}{\sqrt{2}}$. Since, for instance, $\frac{1}{2} = \alpha'_0 \beta'_1 \neq \gamma_{01} = 0$, we conclude quickly that the state in question is not separable. ▲

This paper has shown a computably easy and straightforward routine to test whether quantum states of some special form are separable (inner quantum states). Unlike a system of equations that usually needs to be solved for separability in a tricky way, this one involves only some mechanical calculations.

**Example 6**

Let a quantum state $|\boldsymbol{\varphi}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. We are looking for two states $|\varphi_A\rangle = a_0|0\rangle + +a_1|1\rangle$ and $|\varphi_B\rangle = b_0|0\rangle + b_1|1\rangle$ such that $|\boldsymbol{\varphi}\rangle = |\varphi_A\rangle|\varphi_B\rangle = [a_0b_0; a_0b_1; a_1b_0; a_1b_1]^T$. This gives the following system of equations:

$$\begin{cases} a_0b_0 = \frac{1}{\sqrt{2}} \\ a_0b_1 = 0 \\ a_1b_0 = 0 \\ a_1b_1 = \frac{1}{\sqrt{2}} \end{cases}$$

The reasoning about the solution for the above system requires some idea as this is not a "mechanical" process in any way. That is, there exists no direct formula to decide whether it has any solution or not and each particular case needs specific deduction. Though, the first of the above equations implies that, either $a_0 \neq 0$ or $b_0 \neq 0$. If the former takes place, then the second equation in turn implies that $b_1 = 0$, but this contradicts what results from the fourth equation. Then, maybe, $b_0 \neq 0$, but if so, then the third equation implies that $a_1 = 0$, which again contradicts what results from the fourth equation. We conclude that there is no solution for this system of equations and hence the state $|\boldsymbol{\varphi}\rangle$ is not separable. ▲

**Example 7**

Let a quantum state $|\boldsymbol{\varphi}\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + + \frac{1}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle$. Again, we are looking for two states $|\varphi_A\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\varphi_B\rangle = = b_0|0\rangle + b_1|1\rangle$ such that $|\boldsymbol{\varphi}\rangle = |\varphi_A\rangle|\varphi_B\rangle = = [a_0b_0; a_0b_1; a_1b_0; a_1b_1]^T$, thus giving us the system:

$$\begin{cases} a_0b_0 = \frac{1}{\sqrt{3}} \\ a_0b_1 = \frac{1}{\sqrt{6}} \\ a_1b_0 = \frac{1}{\sqrt{6}} \\ a_1b_1 = \frac{1}{\sqrt{3}} \end{cases}$$

Since $a_0b_0 \neq 0$ and $a_1b_1 \neq 0$ we conclude that $a_0, a_1, b_0, b_1 \neq 0$. From the first two equations we derive $\frac{b_0}{b_1} = \frac{a_0b_0}{a_0b_1} = \frac{\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{6}}} = \sqrt{2}$. From the last two ones we derive $\frac{b_0}{b_1} = \frac{a_1b_0}{a_1b_1} = \frac{\frac{1}{\sqrt{6}}}{\frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{2}} \neq \sqrt{2}$. This contradiction shows that state $|\boldsymbol{\varphi}\rangle$ cannot be separable. ▲

The last two examples show that sometimes a particular case of a system of equations needs a specific reasoning about its solution. In essence, there is no general "prescription" on how to solve such systems. However, the formulas shown in this paper can give a tool to check for separability in a simple way.

Interested reader is referred for further reading on Schmidt decomposition to [1], [2], [4], [6], and in particular to [3], where the links between Schmidt decomposition and singular value decomposition were shown. The latter is described itself in [7]. These two decompositions are more sophisticated tools that would help testing the quantum state entanglement.

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# O pewnych przypadkach szczególnych stanu układu kwantowego w testowaniu jego rozkładalności

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Artykuł prezentuje prosty algorytm obliczeniowy na określanie, czy dany stan kwantowy, w pewnej szczególnej postaci, jest rozkładalny wg zadanego podziału zbioru kubitów na dwie części. Tak więc, mając podzielony zbiór kubitów na dwa, odpowiadamy na pytanie: czy oryginalny stan kwantowy ma postać rozkładalną jako iloczyn tensorowy pewnych dwóch innych stanów kwantowych, które zostały utworzone w oparciu o kubity z każdego z dwóch ww. podzbiorów?

**Słowa kluczowe:** obliczenia kwantowe, rejestr kwantowy, splątanie kwantowe, rozkładalność stanów kwantowych.