

# Modelling quantum register disentanglement

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Implementing quantum-inspired algorithms on classical computers suffers trade-off between the necessity of saving operational memory and the amount of memory necessary to fully represent a quantum state with possible entanglement. The latter is well known to consume the memory exponentially in the number of qubits. This paper sketches out the idea on how to reduce significantly the amount of necessary memory while distorting the entanglement moderately or not at all. At present, considered are real nonnegative probability amplitudes.

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## 1. Introduction

Quantum computing is known to accelerate the process of computation even exponentially in time (see e.g. [2], [3], [4]). However, the quantum computer is in its early stage of construction. On the other hand, the idea of quantum computation gave birth to a variety of quantum-inspired algorithms (see e.g. [6]).

Quantum-inspired algorithms refer to the model of quantum register. It needs, though, exponentially many variables with the number of qubits to define its state. This would be too much for a classical computer to model it efficiently. One may ask whether the amount of data necessary to model could be reduced while still giving a good approximation of the original quantum state. The answer must consist of two points. First, it must give some measure of such approximation. Second, it must give some algorithm of “reduction”. This paper presents both. The reduction consists in splitting the original register into two, thus losing some information due to the possible disentanglement, but decreasing significantly the number of variables necessary to define its state.

## 2. Savings on splitting a quantum register

Assume we have a quantum register of  $n$  qubits. Its state is expressed with  $N = 2^n$  quantum probability amplitudes, say  $\alpha_0, \alpha_1, \dots, \alpha_{N-1} \in \mathbb{C}$ :

$$|\varphi\rangle = \alpha_0|00 \dots 00\rangle + \alpha_1|00 \dots 01\rangle + \dots + \alpha_{N-1}|11 \dots 11\rangle, \quad (1)$$

holding also:

$$|\alpha_0|^2 + |\alpha_1|^2 + \dots + |\alpha_{N-1}|^2 = 1. \quad (2)$$

When confining each  $\alpha_i$  to the real domain  $\mathbb{R}$ , the amount of necessary data can be reduced further by 1 variable. Indeed, without loss the generality, the last variable  $\alpha_{N-1}$  would be computed from the  $N - 1$  previous ones:

$$\alpha_{N-1} = \sqrt{1 - \alpha_0^2 - \alpha_1^2 - \dots - \alpha_{N-2}^2}. \quad (3)$$

From now on, it will be assumed in this paper that the probability amplitudes are nonnegative real, i.e.  $\alpha_i \in [0; 1]$ , for each  $i = 0, 1, \dots, N - 1$ .

Splitting the quantum register of  $n$  qubits into two – one of  $d$  qubits and another one of  $(n - d)$  qubits – would allow to save some number of quantum probability amplitudes. The number of necessary variables for modelling a quantum register of  $d$  qubits is  $2^d - 1$  and, for modelling a quantum register of  $(n - d)$  qubits,  $2^{n-d} - 1$  variables are necessary. Thus, the overall saving is:

$$\begin{aligned} s_{n,d} &= (2^n - 1) - (2^{n-d} - 1) - (2^d - 1) = \\ &= 2^n - 2^{n-d} - 2^d + 1. \end{aligned} \quad (4)$$

Without the loss of generality, it can be assumed that  $d \leq \frac{1}{2}n$  and that the split routine separates the first  $d$  qubits from the other  $(n - d)$  ones.

In fact, if it had not been the case, some appropriate swaps on quantum registers would be applied before and after the presented routine. Then, it is easily seen that the saving is effective for  $1 \leq d \leq \frac{1}{2}n$ :

$$\begin{aligned} s_{n,d} &= (2^{n-d} - 2^{n-2d} - 1)2^d + 1 = \\ &= (2^{n-d}(1 - 2^{-d}) - 1)2^d + 1 \geq \\ &\geq (2^d(1 - 2^{-d}) - 1)2^d + 1 = \\ &= (2^d - 2)2^d + 1 \geq 1. \end{aligned} \quad (5)$$

The above formula shows that any split reduces the amount of data necessary to represent it, i.e., the number of degrees of freedom. Nevertheless, it must be taken into account that the saving on the degrees of freedom would be made usually at the exchange of accuracy loss. That is, any state  $|\varphi\rangle$  can be expressed as per (1), but, due to possible quantum entanglement, is not necessarily separable. It means that not all such states can be expressed as  $|\varphi\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$ , where  $|\varphi_A\rangle = \alpha_0^A|0 \dots 00\rangle + \alpha_1^A|0 \dots 01\rangle + \dots + \alpha_{2^d-1}^A|1 \dots 11\rangle$  and  $|\varphi_B\rangle = \alpha_0^B|0 \dots 00\rangle + \alpha_1^B|0 \dots 01\rangle + \dots + \alpha_{2^{n-d}-1}^B|1 \dots 11\rangle$ , see e.g. [10]. However, every state  $|\varphi\rangle$ , whether separable or not, belongs to a Hilbert space, that has its own norm, which is a continuous function of the coefficients  $\alpha_0, \alpha_1, \dots, \alpha_{2^n-1}$  in (1), and a metric induced by this norm (see e.g. [6], [7]). Assume that there exist a transformation  $\mathcal{T}$  on the space of quantum states, continuous with the above metric and that, for every state as an argument, gives a separable state as its value. Assume also that, for every separable state,  $\mathcal{T}$  is invariant, i.e. gives this same state. Intuitively,  $\mathcal{T}|\varphi\rangle = |\varphi\rangle$ , if  $|\varphi\rangle$  is separable, and  $\mathcal{T}|\varphi\rangle$  is close to  $|\varphi\rangle$ , if  $|\varphi\rangle$  is “almost separable”. This is due to the assumed continuity of  $\mathcal{T}$ . Then, the distance between  $\mathcal{T}|\varphi\rangle$  and  $|\varphi\rangle$  would be a candidate for measuring the strength of entanglement. The  $\mathcal{T}$ , in turn, would be called a **disentanglement operator**.

### 3. Disentanglement measure

Assume, like in the previous section, that the quantum register is to be split so that the first  $d$  qubits would form one register and the next  $(n-d)$  would form the other one. The classical state  $|\varphi\rangle \in \mathbb{H}^{\otimes n} = \mathbb{H}^{\otimes d} \otimes \mathbb{H}^{\otimes(n-d)}$  of the original quantum register can be expressed as

$$|\varphi\rangle = \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl} |k\rangle |l\rangle, \quad (6)$$

where each  $0 \leq \gamma_{kl} \leq 1$  (is nonnegative real). Meanwhile, to separate it between  $\mathbb{H}^{\otimes d}$  and  $\mathbb{H}^{\otimes(n-d)}$ , a new state  $\mathcal{T}|\varphi\rangle = |\varphi'\rangle = |\varphi'_A\rangle |\varphi'_B\rangle$  is necessary,  $|\varphi'_A\rangle \in \mathbb{H}^{\otimes d}$ ,  $|\varphi'_B\rangle \in \mathbb{H}^{\otimes(n-d)}$ . The data capacity saving, as stated in the previous section, is made because the number of quantum probability amplitudes of  $|\varphi'_A\rangle$  and  $|\varphi'_B\rangle$  is lower than that of the original  $|\varphi\rangle$ . The data loss, due to such a split (transformation  $\mathcal{T}$ ), might be expressed as a distance between  $|\varphi\rangle$  and  $\mathcal{T}|\varphi\rangle = |\varphi'\rangle$ . Since both are of the norm 1, we obtain that

$$\begin{aligned} \|\varphi' - \varphi\|^2 &= \langle \varphi' - \varphi | \varphi' - \varphi \rangle = \\ &= \langle \varphi' | \varphi' \rangle - \langle \varphi' | \varphi \rangle - \langle \varphi | \varphi' \rangle + \langle \varphi | \varphi \rangle = \\ &= \langle \varphi | \varphi \rangle + \langle \varphi' | \varphi' \rangle - \langle \varphi | \varphi' \rangle - \langle \varphi | \varphi' \rangle = \\ &= 2 - 2\text{Re}\{\langle \varphi | \varphi' \rangle\}. \end{aligned} \quad (7)$$

Since  $|\varphi\rangle$  is assumed to have real coefficients, the following formula for the distance is derived:

$$d(\varphi, \varphi') = \|\varphi' - \varphi\| = \sqrt{2(1 - \langle \varphi | \varphi' \rangle)}. \quad (8)$$

The distance  $d$  ranges from 0, for  $|\varphi\rangle = |\varphi'\rangle$ , to 2, for  $|\varphi\rangle = -|\varphi'\rangle$ . Thus, 0 means that  $|\varphi\rangle$  is separable, whereas the greater the measure  $d$ , the more entanglement is presumed. Now, note that the function  $f: x \rightarrow \left(1 - \frac{x^2}{2}\right)$ ,  $0 \leq x \leq 2$ , is monotone decreasing,  $f(0) = 1$  and  $f(2) = -1$ . To simplify the things a little bit, let us define the similarity (denoted by  $\varepsilon$ ) as

$$\varepsilon = f(d(\varphi, \varphi')) = \langle \varphi | \varphi' \rangle. \quad (9)$$

This is the cosine of the angle between the state  $|\varphi\rangle$  and  $|\varphi'\rangle$ . It is easily seen that, if  $|\varphi'\rangle = |\varphi\rangle$ , the similarity is  $\varepsilon = 1$ . Hence, it is required that  $|\varphi'_A\rangle$  and  $|\varphi'_B\rangle$  are constructed in such a way that, for  $|\varphi\rangle = |\varphi'\rangle = |\varphi'_A\rangle \otimes |\varphi'_B\rangle$ , the similarity measure is  $\langle \varphi | \varphi' \rangle = 1$ .

Now, let the disentanglement operator  $\mathcal{T}$  be such that:

$$|\varphi'_A\rangle = \left( \sum_{k=0}^{2^d-1} A_k |k\rangle \right) \quad (10)$$

and

$$|\varphi'_B\rangle = \left( \sum_{l=0}^{2^{(n-d)}-1} B_l |l\rangle \right), \quad (11)$$

where

$$A_k = \sqrt{\sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl}^2} \quad (12)$$

and

$$B_l = \sqrt{\sum_{k=0}^{2^d-1} \gamma_{kl}^2}. \quad (13)$$

The coefficients (12) and (13) define correct  $|\varphi'_A\rangle$  and  $|\varphi'_B\rangle$ . Indeed, the squares of coefficients for  $|\varphi'_A\rangle$  sum up to

$$\begin{aligned} \sum_{k=0}^{2^d-1} A_k^2 &= \sum_{k=0}^{2^d-1} \left( \sqrt{\sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl}^2} \right)^2 = \\ &= \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl}^2 = 1. \end{aligned} \quad (14)$$

Likewise, the squares of coefficients for  $|\varphi'_B\rangle$  sum up to

$$\begin{aligned} \sum_{l=0}^{2^{(n-d)}-1} B_l^2 &= \sum_{l=0}^{2^{(n-d)}-1} \left( \sqrt{\sum_{k=0}^{2^d-1} \gamma_{kl}^2} \right)^2 = \\ &= \sum_{l=0}^{2^{(n-d)}-1} \sum_{k=0}^{2^d-1} \gamma_{kl}^2 = 1. \end{aligned} \quad (15)$$

The above two formulas are true in view of how  $|\varphi\rangle$  is defined in (6). The squares of its coefficients must sum to 1, so do those in (14) and (15).

Having defined the separation formula (10), (11), let us check its impact on the original state. The measure  $\varepsilon$  gives the following inner product of the two:

$$\langle \varphi | \varphi' \rangle = \langle \varphi | (|\varphi'_A\rangle |\varphi'_B\rangle) =$$

$$\begin{aligned} &= \left( \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl} \langle k | \langle l | \right) \cdot \\ &\cdot \left( \sum_{m=0}^{2^d-1} \sum_{n=0}^{2^{(n-d)}-1} A_m B_n |m\rangle |n\rangle \right) = \\ &= \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \sum_{m=0}^{2^d-1} \sum_{n=0}^{2^{(n-d)}-1} \gamma_{kl} A_m B_n \langle k | \langle l | |m\rangle |n\rangle = \\ &= \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \sum_{m=0}^{2^d-1} \sum_{n=0}^{2^{(n-d)}-1} \gamma_{kl} A_m B_n \langle k | m \rangle \langle l | n \rangle = \\ &= \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl} A_k B_l. \end{aligned} \quad (16)$$

Please note, that, for a separable state  $|\varphi\rangle$  as per (6), the conversion (10) – (13) is ineffective. That is, for a state  $|\varphi\rangle = |\varphi_A\rangle |\varphi_B\rangle$ , where  $|\varphi_A\rangle \in \mathbb{H}^{\otimes d}$  and  $|\varphi_B\rangle \in \mathbb{H}^{\otimes (n-d)}$ , the resulting state  $|\varphi'\rangle = |\varphi'_A\rangle |\varphi'_B\rangle$ , where  $|\varphi'_A\rangle \in \mathbb{H}^{\otimes d}$  and  $|\varphi'_B\rangle \in \mathbb{H}^{\otimes (n-d)}$ , is the same and, of course,  $\langle \varphi | \varphi' \rangle = 1$ . Indeed, assuming that

$$|\varphi_A\rangle = \sum_{k=0}^{2^d-1} \alpha_k |k\rangle \quad (17)$$

and

$$|\varphi_B\rangle = \sum_{l=0}^{2^{(n-d)}-1} \beta_l |l\rangle, \quad (18)$$

where, of course,

$$\sum_{k=0}^{2^d-1} \alpha_k^2 = \sum_{l=0}^{2^{(n-d)}-1} \beta_l^2 = 1, \quad (19)$$

we have

$$\begin{aligned} |\varphi\rangle &= \left( \sum_{k=0}^{2^d-1} \alpha_k |k\rangle \right) \left( \sum_{l=0}^{2^{(n-d)}-1} \beta_l |l\rangle \right) = \\ &= \sum_{k=0}^{2^d-1} \sum_{l=0}^{2^{(n-d)}-1} \alpha_k \beta_l |k\rangle |l\rangle. \end{aligned} \quad (20)$$

Hence,  $\gamma_{kl} = \alpha_k \beta_l$  in (6) and

$$\begin{aligned}
 A_k &= \sqrt{\sum_{l=0}^{2^{(n-d)}-1} \gamma_{kl}^2} = \sqrt{\sum_{l=0}^{2^{(n-d)}-1} \alpha_k^2 \beta_l^2} = \\
 &= \alpha_k \sqrt{\sum_{l=0}^{2^{(n-d)}-1} \beta_l^2} = \alpha_k.
 \end{aligned} \tag{21}$$

Similarly,

$$\begin{aligned}
 B_l &= \sqrt{\sum_{k=0}^{2^d-1} \gamma_{kl}^2} = \sqrt{\sum_{k=0}^{2^d-1} \alpha_k^2 \beta_l^2} = \\
 &= \beta_l \sqrt{\sum_{k=0}^{2^d-1} \alpha_k^2} = \beta_l.
 \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
 \langle \varphi | \varphi' \rangle &= (\langle \varphi_A | \langle \varphi_B |) (\langle \varphi'_A | \langle \varphi'_B |) = \\
 &= \langle \varphi_A | \varphi'_A \rangle \langle \varphi_B | \varphi'_B \rangle = \\
 &= \left( \sum_{k=0}^{2^d-1} \alpha_k \langle k | \right) \left( \sum_{m=0}^{2^d-1} A_m |m\rangle \right) \cdot \\
 &\quad \cdot \left( \sum_{l=0}^{2^{(n-d)}-1} \beta_l \langle l | \right) \left( \sum_{n=0}^{2^{(n-d)}-1} B_n |n\rangle \right) = \\
 &= \left( \sum_{k=0}^{2^d-1} \alpha_k A_k \right) \cdot \left( \sum_{l=0}^{2^{(n-d)}-1} \beta_l B_l \right) = \\
 &= \left( \sum_{k=0}^{2^d-1} \alpha_k^2 \right) \cdot \left( \sum_{l=0}^{2^{(n-d)}-1} \beta_l^2 \right) = 1.
 \end{aligned} \tag{23}$$

Both  $|\varphi\rangle$  and  $|\varphi'\rangle$  have nonnegative real coefficients in the linear combination of basic states expressing them. So do their Hermitian adjoints  $\langle\varphi|$  and  $\langle\varphi'|$ , which are mere transpositions in this case. Thus, (23) means that  $|\varphi\rangle = |\varphi'\rangle$ . ■

#### 4. Entanglement measure

The decision on how to divide a quantum register into two parts must be based on how the qubits are entangled internally. That is, it must be detected somehow which qubit

observation would affect subsequent observations of other qubits.

In quantum physics, the entanglement between two qubits results in correlation between their observations. The observation of two qubits in the standard basis, whether done consecutively or simultaneously, gives its outcomes with some probabilities. It is simple to show that these two measurement schemes on a state  $|q_n q_{n-1} \dots q_1\rangle$ , where each  $q_k$  is 0 or 1, are equivalent:

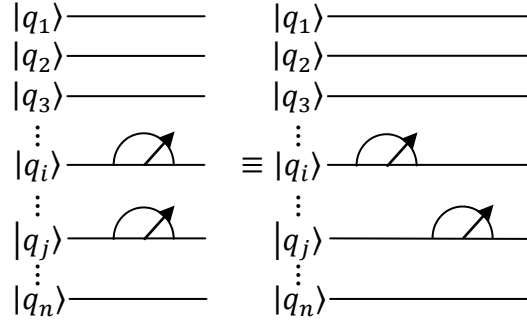


Fig. 1. Simultaneous and consecutive measurement of two qubits

Without loss of generality, it can be assumed that  $i = n$  and  $j = n - 1$ . If this is not the case, appropriate swap operations  $S$  and  $S^{-1}$  can be applied:

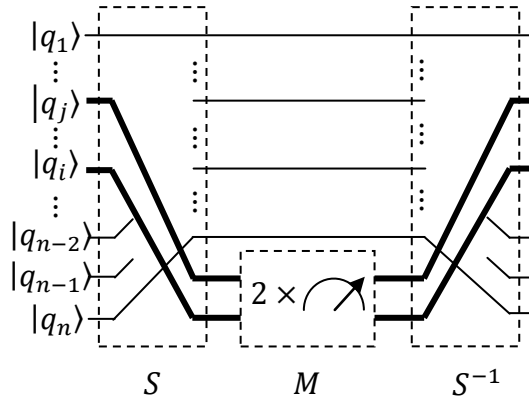


Fig. 2. Swap operations before and after measurement

Thus, it is sufficient to show that simultaneous and consecutive measurement schemes are equivalent for the two leftmost qubits. The operation  $M$  in figure 2 is one of such non-unitary measurements. The following lemma proves the equivalence of the two schemes:

#### Lemma 1

Two consecutive measurements (in the standard basis) on the leftmost and the second leftmost qubits result in the outcomes with the same

probabilities as if these two were measured simultaneously.

**Proof**

Let us consider the simultaneous measurement scheme first. Let  $|\varphi\rangle$  be an initial state as per (1). Let us denote the event of an outcome on the most significant  $m$  qubits<sup>1</sup> after a simultaneous measurement being  $|x_m\rangle|x_{m-1}\rangle \dots |x_1\rangle$ ,  $x_k \in \{0,1,*\}$  for  $k = 1, 2, \dots, m$ , by “ $x_m x_{m-1} \dots x_1 \dots$ ” (the star meaning “0 or 1”). Let us also denote the event of the same outcome, but after consecutive measurements, by “ $x_m \dots x_{i+1}, x_i \dots x_1 \dots$ ”, i.e. the comma separates the qubit outcomes measured at different times. The probabilities of outcomes  $|0\rangle|0\rangle$ ,  $|0\rangle|1\rangle$ ,  $|1\rangle|0\rangle$  and  $|1\rangle|1\rangle$  on the two leftmost qubits after measuring them simultaneously are as follows:

$$\begin{aligned} \Pr\{00 \dots\} &= |\alpha_0|^2 + |\alpha_1|^2 + \dots + \left| \alpha_{\frac{N}{4}-1} \right|^2, \\ \Pr\{01 \dots\} &= \left| \alpha_{\frac{N}{4}} \right|^2 + \left| \alpha_{\frac{N}{4}+1} \right|^2 + \dots + \left| \alpha_{\frac{N}{2}-1} \right|^2, \\ \Pr\{10 \dots\} &= \left| \alpha_{\frac{N}{2}} \right|^2 + \left| \alpha_{\frac{N}{2}+1} \right|^2 + \dots + \left| \alpha_{\frac{3N}{4}-1} \right|^2, \\ \Pr\{11 \dots\} &= \left| \alpha_{\frac{3N}{4}} \right|^2 + \left| \alpha_{\frac{3N}{4}+1} \right|^2 + \dots + |\alpha_{N-1}|^2. \end{aligned} \quad (24)$$

On the other hand, when measuring only the leftmost qubit, the outcome probabilities are:

$$\begin{aligned} \Pr\{0 \dots\} &= |\alpha_0|^2 + |\alpha_1|^2 + \dots + \left| \alpha_{\frac{N}{2}-1} \right|^2, \\ \Pr\{1 \dots\} &= \left| \alpha_{\frac{N}{2}} \right|^2 + \left| \alpha_{\frac{N}{2}+1} \right|^2 + \dots + |\alpha_{N-1}|^2, \end{aligned} \quad (25)$$

and the quantum states after the collapse are, respectively,

$$\begin{aligned} |\varphi_0\rangle &= c_0 \left( \alpha_0 |00 \dots 00\rangle + \alpha_1 |00 \dots 01\rangle + \dots \right. \\ &\quad \left. + \alpha_{\frac{N}{2}-1} |01 \dots 11\rangle \right), \\ |\varphi_1\rangle &= c_1 \left( \alpha_{\frac{N}{2}} |10 \dots 00\rangle + \alpha_{\frac{N}{2}+1} |10 \dots 01\rangle + \dots \right. \\ &\quad \left. + \alpha_{N-1} |11 \dots 11\rangle \right), \end{aligned} \quad (26)$$

where  $c_0, c_1$  are normalising coefficients:

<sup>1</sup> These qubits are the leftmost ones in the Dirac’s ket notation and are located at the bottom when depicted in a quantum circuit.

$$\begin{aligned} c_0 &= \frac{1}{\sqrt{|\alpha_0|^2 + |\alpha_1|^2 + \dots + \left| \alpha_{\frac{N}{2}-1} \right|^2}} = \\ &= \frac{1}{\sqrt{\Pr\{0 \dots\}}}, \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{1}{\sqrt{\left| \alpha_{\frac{N}{2}} \right|^2 + \left| \alpha_{\frac{N}{2}+1} \right|^2 + \dots + |\alpha_{N-1}|^2}} = \\ &= \frac{1}{\sqrt{\Pr\{1 \dots\}}}. \end{aligned} \quad (27)$$

Note that  $\Pr\{0 \dots\} = \Pr\{00 \dots\} + \Pr\{01 \dots\}$  and  $\Pr\{1 \dots\} = \Pr\{10 \dots\} + \Pr\{11 \dots\}$ . After measuring the second leftmost qubit, provided that the state of the leftmost one was  $|x\rangle$  before the second measurement, an outcome state  $|x\rangle|y\rangle$  on the two qubits in question is described by the conditional probability:

$$\begin{aligned} \Pr\{x, y \dots | x \dots\} &= \Pr\{x, y \dots \cap x \dots\} / \Pr\{x \dots\} \\ &= \Pr\{x, y \dots\} / \Pr\{x \dots\}, \end{aligned} \quad (28)$$

for  $x$  and  $y$  from  $\{0,1\}$ . That is, the probability of observing  $y$  on the second leftmost qubit after  $x$  has been observed in the leftmost one, is as per (28). This gives:

$$\Pr\{x, y \dots\} = \Pr\{x, y \dots | x \dots\} \cdot \Pr\{x \dots\} \quad (29)$$

Next,

$$\begin{aligned} \Pr\{0,0 \dots | 0 \dots\} &= \\ &= |c_0 \alpha_0|^2 + |c_0 \alpha_1|^2 + \dots + \left| c_0 \alpha_{\frac{N}{4}-1} \right|^2 \\ &= \frac{1}{\Pr\{0 \dots\}} \left( |\alpha_0|^2 + |\alpha_1|^2 + \dots + \left| \alpha_{\frac{N}{4}-1} \right|^2 \right), \end{aligned}$$

$$\begin{aligned} \Pr\{0,1 \dots | 0 \dots\} &= \\ &= \left| c_0 \alpha_{\frac{N}{4}} \right|^2 + \left| c_0 \alpha_{\frac{N}{4}+1} \right|^2 + \dots + \left| c_0 \alpha_{\frac{N}{2}-1} \right|^2 \\ &= \frac{1}{\Pr\{0 \dots\}} \left( \left| \alpha_{\frac{N}{4}} \right|^2 + \left| \alpha_{\frac{N}{4}+1} \right|^2 + \dots + \left| \alpha_{\frac{N}{2}-1} \right|^2 \right), \end{aligned}$$

$$\begin{aligned} \Pr\{1,0 \dots | 1 \dots\} &= \\ &= \left| c_1 \alpha_{\frac{N}{2}} \right|^2 + \left| c_1 \alpha_{\frac{N}{2}+1} \right|^2 + \dots + \left| c_1 \alpha_{\frac{3N}{4}-1} \right|^2 = \\ &= \frac{1}{\Pr\{1 \dots\}} \left( \left| \alpha_{\frac{N}{2}} \right|^2 + \left| \alpha_{\frac{N}{2}+1} \right|^2 + \dots + \left| \alpha_{\frac{3N}{4}-1} \right|^2 \right), \end{aligned}$$

$$\begin{aligned}
 \Pr\{1,1 \dots |1 \dots\} &= \\
 &= \left|c_1 \alpha_{\frac{3N}{4}}\right|^2 + \left|c_1 \alpha_{\frac{3N}{4}+1}\right|^2 + \dots + |c_1 \alpha_{N-1}|^2 = \\
 &= \frac{1}{\Pr\{1 \dots\}} \left( \left| \alpha_{\frac{3N}{4}} \right|^2 + \left| \alpha_{\frac{3N}{4}+1} \right|^2 + \dots \right. \\
 &\quad \left. + |\alpha_{N-1}|^2 \right). \tag{30}
 \end{aligned}$$

Finally, by (29) and (30):

$$\begin{aligned}
 \Pr\{0,0 \dots\} &= |\alpha_0|^2 + |\alpha_1|^2 + \dots + \left| \alpha_{\frac{N}{4}-1} \right|^2 = \\
 &= \Pr\{00 \dots\}, \\
 \Pr\{0,1 \dots\} &= \left| \alpha_{\frac{N}{4}} \right|^2 + \left| \alpha_{\frac{N}{4}+1} \right|^2 + \dots + \left| \alpha_{\frac{N}{2}-1} \right|^2 = \\
 &= \Pr\{01 \dots\}, \\
 \Pr\{1,0 \dots\} &= \left| \alpha_{\frac{N}{2}} \right|^2 + \left| \alpha_{\frac{N}{2}+1} \right|^2 + \dots \\
 &\quad + \left| \alpha_{\frac{3N}{4}-1} \right|^2 = \Pr\{10 \dots\}, \\
 \Pr\{1,1 \dots\} &= \left| \alpha_{\frac{3N}{4}} \right|^2 + \left| \alpha_{\frac{3N}{4}+1} \right|^2 + \dots \\
 &\quad + |\alpha_{N-1}|^2 = \Pr\{11\}. \tag{31}
 \end{aligned}$$

This proves that consecutive measurements in the standard basis give their outcomes with probabilities  $\Pr\{x,y \dots\}$  equal to the corresponding probabilities  $\Pr\{xy \dots\}$  of outcomes in simultaneous measurement. ■

This simplified approach of measuring the entanglement only within pairs of qubits is designed for the purposes of quantum register splitting algorithm (developed later in this paper). However, one should bear in mind that the entanglement is a wider phenomenon and might involve more than 2 qubits. For example, no measurement on the leftmost or the central qubit in the state  $|\varphi\rangle = \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$  can determine the state of the rightmost one. Whether the outcome is  $|0\rangle$  or  $|1\rangle$ , the rightmost qubit always remains  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . On the other hand, when both, the leftmost and central qubits, are measured together, their outcomes determine the state of the rightmost one –  $|00\rangle$  and  $|11\rangle$  determine that this is  $|1\rangle$ , whereas  $|01\rangle$  and  $|10\rangle$  determine that this is  $|0\rangle$ . This shows that, in terms of

entanglement, the 3 qubits should be considered together in the above state.

Now, to easily measure the strength of entanglement between two qubits, say  $i$ 'th and  $j$ 'th ones, it is necessary to determine the possible outcomes of measurements on them. Knowing the current quantum state, the measurement of  $i$ -th and  $j$ -th qubits can be simulated. As just shown, it is indifferent for this experiment whether the measurement takes place in the consecutive or simultaneous scheme. Let  $\ell_k(a)$  denote the  $k$ -th least significant digit in the binary representation of  $a$  (or the digit corresponding to the  $k$ -th qubit from the right). Then, the probability  $p_{ij}^{xy}$  of the  $i$ -th and  $j$ -th qubits collapse after measurement to  $|x\rangle$  and  $|y\rangle$ , respectively, where  $x, y \in \{0,1\}$ , is

$$p_{ij}^{xy} = \sum_{\substack{\ell_i(a)=x, \\ \ell_j(a)=y}} |\alpha_a|^2, \tag{32}$$

where  $\alpha_a$ , for  $a = 0, 1, \dots, N-1$ , as per (1).

### Example 1

Let  $|\varphi\rangle = \alpha_1|001\rangle + \alpha_3|011\rangle + \alpha_7|111\rangle$ , where  $\alpha_1 = \alpha_3 = \alpha_7 = \frac{1}{\sqrt{3}}$ . For the first two qubits from the right, the following outcome probabilities are derived:  $p_{12}^{00} = p_{12}^{01} = 0$ ,  $p_{12}^{10} = |\alpha_1|^2 = \frac{1}{3}$ ,  $p_{12}^{11} = |\alpha_3|^2 + |\alpha_7|^2 = \frac{2}{3}$ . The second and the third qubits result in:  $p_{23}^{00} = |\alpha_1|^2 = \frac{1}{3}$ ,  $p_{23}^{01} = 0$ ,  $p_{23}^{10} = |\alpha_3|^2 = \frac{1}{3}$ ,  $p_{23}^{11} = |\alpha_7|^2 = \frac{1}{3}$ , and the pair of the first and the third ones result in  $p_{13}^{00} = p_{13}^{01} = 0$ ,  $p_{13}^{10} = |\alpha_1|^2 + |\alpha_3|^2 = \frac{2}{3}$ ,  $p_{13}^{11} = |\alpha_7|^2 = \frac{1}{3}$ . ▲

The outcomes on the two qubits in question, along with their probabilities, can be considered as two independent random variables  $Z_i$  and  $Z_j$  taking binary values. Thus, their 2-dimensional joint probability distribution, two marginal probability distributions, and Pearson correlation coefficient  $\rho_{ij}$  might be computed. The following table shows the aforementioned probability distributions:

$Z_j \backslash Z_i$	0	1	$\Sigma$
0	$p_{ij}^{00}$	$p_{ij}^{01}$	$p_{ij}^{0\cdot} = p_{ij}^{00} + p_{ij}^{01}$
1	$p_{ij}^{10}$	$p_{ij}^{11}$	$p_{ij}^{1\cdot} = p_{ij}^{10} + p_{ij}^{11}$
$\Sigma$	$p_{ij}^{\cdot 0} = p_{ij}^{00} + p_{ij}^{10}$	$p_{ij}^{\cdot 1} = p_{ij}^{01} + p_{ij}^{11}$	$p_{ij}^{\cdot\cdot} = \sum_{a,b} p_{ij}^{ab} = 1$

 Tab. 1. Probability distributions for computing  $\varrho_{ij}$ 

Note that all the powers of a binary value are equal, i.e.  $0 = 0^k$  and  $1 = 1^k$ , for all  $k \in \mathbb{N}_+$ . Hence, the  $k$ -th power of some binary random variable  $Z$  have the same probability distribution as the underlying random variable itself. This implies that

$$E[Z^k] = E[Z], \quad (33)$$

where  $Z$  – binary distributed (takes value of 0 or 1).

Recall that the Pearson correlation coefficient is computed as follows (see e.g. [1], [5], [9]):

$$\varrho_{ij} = \frac{\text{Cov}(Z_i, Z_j)}{DZ_i \cdot DZ_j}, \quad (34)$$

where  $DZ_i$  and  $DZ_j$  are the standard deviations of  $Z_i$  and  $Z_j$ , respectively, and  $\text{Cov}(Z_i, Z_j)$  is the covariance between  $Z_i$  and  $Z_j$ . The covariance is, in turn, computed as (see e.g. [1], [5], [9])

$$\begin{aligned} \text{Cov}(Z_i, Z_j) &= E[(Z_i - EZ_i)(Z_j - EZ_j)] = \\ &= E[Z_i Z_j] - EZ_i \cdot EZ_j. \end{aligned} \quad (35)$$

and the variance (see e.g. [1], [9])

$$D^2 Z = E[Z - EZ]^2 = E[Z^2] - E^2[Z]. \quad (36)$$

Due to (33), for binary random variables, we obtain

$$D^2 Z = EZ - E^2[Z] = EZ(1 - EZ). \quad (37)$$

Further,

$$EZ_i = E[Z_i]^2 = \frac{1}{p_{ij}^{\cdot\cdot}} (0 \cdot p_{ij}^{0\cdot} + 1 \cdot p_{ij}^{1\cdot}) = p_{ij}^{1\cdot}, \quad (38)$$

and, similarly,

$$EZ_j = E[Z_j]^2 = \frac{1}{p_{ij}^{\cdot\cdot}} (0 \cdot p_{ij}^{\cdot 0} + 1 \cdot p_{ij}^{\cdot 1}) = p_{ij}^{\cdot 1}. \quad (39)$$

Next, the expectation of the product is

$$E[Z_i Z_j] = \frac{1}{p_{ij}^{\cdot\cdot}} \left( \sum_{a,b \in \{0,1\}} a \cdot b \cdot p_{ij}^{ab} \right) = p_{ij}^{11}. \quad (40)$$

Thus, the covariance is

$$\begin{aligned} \text{Cov}(Z_i, Z_j) &= p_{ij}^{11} - p_{ij}^{1\cdot} p_{ij}^{\cdot 1} = p_{ij}^{11} p_{ij}^{\cdot\cdot} - p_{ij}^{1\cdot} p_{ij}^{\cdot 1} = \\ &= p_{ij}^{00} p_{ij}^{11} - p_{ij}^{01} p_{ij}^{10}. \end{aligned} \quad (41)$$

The variance of  $Z_i$  is

$$\begin{aligned} D^2 Z_i &= p_{ij}^{1\cdot} (1 - p_{ij}^{1\cdot}) = p_{ij}^{1\cdot} p_{ij}^{0\cdot} = \\ &= (p_{ij}^{10} + p_{ij}^{11})(p_{ij}^{00} + p_{ij}^{01}). \end{aligned} \quad (42)$$

Likewise, the variance of  $Z_j$  is

$$\begin{aligned} D^2 Z_j &= p_{ij}^{\cdot 1} (1 - p_{ij}^{\cdot 1}) = p_{ij}^{\cdot 1} p_{ij}^{\cdot 0} = \\ &= (p_{ij}^{01} + p_{ij}^{11})(p_{ij}^{00} + p_{ij}^{10}). \end{aligned} \quad (43)$$

Finally,

$$\begin{aligned} \varrho_{ij} &= \frac{p_{ij}^{00} p_{ij}^{11} - p_{ij}^{01} p_{ij}^{10}}{\sqrt{p_{ij}^{1\cdot} p_{ij}^{0\cdot} p_{ij}^{\cdot 1} p_{ij}^{\cdot 0}}} = \\ &= \frac{p_{ij}^{00} p_{ij}^{11} - p_{ij}^{01} p_{ij}^{10}}{\sqrt{(p_{ij}^{10} + p_{ij}^{11})(p_{ij}^{00} + p_{ij}^{01})(p_{ij}^{01} + p_{ij}^{11})(p_{ij}^{00} + p_{ij}^{10})}} \end{aligned} \quad (44)$$

Note that  $\varrho_{ij} = \varrho_{ji}$ . Indeed, since  $p_{ij}^{ab} = p_{ji}^{ba}$ ,

$$\begin{aligned} \varrho_{ji} &= \frac{p_{ji}^{00} p_{ji}^{11} - p_{ji}^{01} p_{ji}^{10}}{\sqrt{p_{ji}^{1\cdot} p_{ji}^{0\cdot} p_{ji}^{\cdot 1} p_{ji}^{\cdot 0}}} = \frac{p_{ij}^{00} p_{ij}^{11} - p_{ij}^{01} p_{ij}^{10}}{\sqrt{p_{ij}^{\cdot 1} p_{ij}^{\cdot 0} p_{ij}^{1\cdot} p_{ij}^{0\cdot}}} = \\ &= \varrho_{ij}. \end{aligned} \quad (45)$$

Of course,  $\varrho_{ii} = 1$ . Since  $p_{ii}^{01} = p_{ii}^{10} = 0$  and assuming  $p_{ij}^{ab} > 0$  we have

$$\varrho_{ii} = \frac{p_{ii}^{00} p_{ii}^{11} - p_{ii}^{01} p_{ii}^{10}}{\sqrt{p_{ii}^{1\cdot} p_{ii}^{0\cdot} p_{ii}^{\cdot 1} p_{ii}^{\cdot 0}}} = \frac{p_{ii}^{00} p_{ii}^{11}}{\sqrt{p_{ij}^{11} p_{ij}^{00} p_{ij}^{11} p_{ij}^{00}}} = 1. \quad (46)$$

The formula (46) shows that even if  $p_{ij}^{ab} \rightarrow 0$ , or is close to 0,  $q_{ii} = 1$ . However, both  $p_{ii}^{00}$  and  $p_{ii}^{11}$  must be nonzero for the above formula to be correct. Otherwise,  $q_{ii}$  is undefined. In general, for  $q_{ij}$  to be defined, all the sums  $p_{ij}^1$ ,  $p_{ij}^0$ ,  $p_{ij}^1$  and  $p_{ij}^0$  must be nonzero. It will be assumed that  $q_{ij}$  is 0, if in fact  $q_{ij}$  is undefined. The reason is that, in such case,  $q_{ij}$  shows no correlation to take into account. All the computed coefficients can be collected in a correlation matrix:

$$\begin{aligned}
 \varrho &= \begin{bmatrix} \varrho_{11} & \varrho_{12} & \cdots & \varrho_{1n} \\ \varrho_{21} & \varrho_{22} & \cdots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{n1} & \varrho_{n2} & \cdots & \varrho_{nn} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & \varrho_{12} & \cdots & \varrho_{1n} \\ \varrho_{12} & 1 & \cdots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{1n} & \varrho_{2n} & \cdots & 1 \end{bmatrix}.
 \end{aligned} \tag{47}$$

### Example 2

Based on the state shown in example 1, the following Pearson correlation coefficients are obtained:  $q_{12} = \frac{0 \cdot \frac{2}{3} - 0 \cdot \frac{1}{3}}{\sqrt{(\frac{1}{3} + \frac{2}{3})(0+0)(0+\frac{2}{3})(0+\frac{1}{3})}}$  is undefined, hence by definition equal to 0, next  $q_{23} = \frac{\frac{1}{3} \cdot 0 - 0 \cdot \frac{1}{3}}{\sqrt{(\frac{1}{3} + \frac{1}{3})(\frac{1}{3} + 0)(0 + \frac{1}{3})(\frac{1}{3} + \frac{1}{3})}} = \frac{1}{2}$ , and finally  $q_{13} = \frac{0 \cdot \frac{1}{3} - 0 \cdot \frac{2}{3}}{\sqrt{(\frac{1}{3} + \frac{2}{3})(0+0)(0+\frac{1}{3})(0+\frac{2}{3})}}$  is also undefined, hence 0. ▲

## 5. Balanced splitting algorithm

The balanced splitting algorithm divides one quantum register into two parts with the numbers of qubits differing at most by 1. It is desired that highly entangled qubits stay in the same part, whereas those “loosely coupled” should be spread across the different parts. Thus, the register would be marginally affected by the disentanglement. Intuitively, this heuristic assumption means that such a split would entail the least possible loss of information.

The level of entanglement between two qubits,  $i$ -th and  $j$ -th, is assessed by the Pearson correlation coefficient  $q_{ij}$ . All such coefficients are gathered together in the correlation matrix  $\varrho = [q_{ij}]_{i,j \in \{1, \dots, n\}}$ . Since  $q_{ij} = q_{ji}$  and  $q_{ii} = 1$ , further considerations will be based on the coefficients from above the diagonal, i.e.  $q_{ij}$  such that  $i < j$ . The first step of the splitting

algorithm is to sort these coefficients in an ascending order.

$$q_{i_1 j_1} \leq q_{i_2 j_2} \leq \cdots \leq q_{i_w j_w}, \tag{48}$$

where

$$w = \frac{n(n-1)}{2}, \tag{49}$$

and, for  $a \neq b$ ,

$$(i_a, j_a) \neq (i_b, j_b). \tag{50}$$

The pair  $(i_1, j_1)$  represents the indices of the least entangled pair of qubits and  $(i_w, j_w)$  of the most entangled one.

The algorithm operates in a clique whose nodes represent qubits and labelled edges represent the level of the entanglement between two neighbouring qubits. The first step of the algorithm is to find the matching in the clique of the minimal overall entanglement. This is done as follows: iterate over the ordered pairs of indices  $(i_1, j_1), (i_2, j_2), \dots, (i_w, j_w)$ . If, for a given pair  $(i_k, j_k)$ , where  $k = 1, 2, \dots, w$ , neither  $i_k$ -th qubit nor  $j_k$ -th one have been selected yet for the resulting matching, add the edge  $(i_k, j_k)$ . If not, try with the next  $k$ . Repeat so, until at least two nodes have not been covered yet by the matching.

### Example 3

Let  $n = 6$  be the number of qubits and the correlation matrix be as follows:

$$\varrho = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{3} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{\sqrt{3}} & 1 \end{bmatrix}.$$

There are different coefficients of the same value, but assume that a sorting algorithm gave the following order:  $q_{13} \leq q_{14} \leq q_{15} \leq q_{16} \leq q_{23} \leq q_{24} \leq q_{25} \leq q_{26} \leq q_{36} \leq q_{46} \leq q_{56} \leq q_{35} \leq q_{45} \leq q_{12} \leq q_{34}$ . 6 nodes are going to represent the qubits, among which a matching is to be found. Recall that each matched pair represent two qubits that must fall into two different subsets after the split. At first, the matching contains no pairs. The algorithm iterates over the pairs (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,6), (4,6), (5,6), (3,5), (4,5), (1,2), (3,4). The first pair (1,3) gives the first pair of the constructed matching, because neither node 1 nor node 3 have been



engaged in there yet. Next, the pairs (1,4), (1,5), (1,6), (2,3) will not be considered as they contain either 1 or 3. The pair (2,4) is found to be the next one in the matching. The pairs (2,6), (3,6), (4,6) are rejected and, finally, (5,6) is selected as the last pair for the matching. Of course, (3,5), (4,5), (1,2), (3,4) must be rejected as there is no node left. The following picture shows the resulting matching:

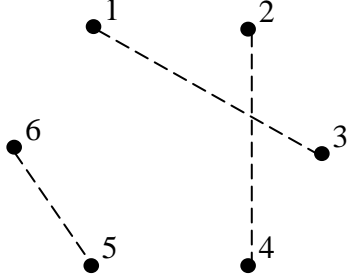


Fig. 3. A matching to separate qubits



The second stage of the splitting algorithm consists in determining two disjoint subsets such that both cover all the nodes but no two nodes from a pair in the matching belong to the same one. It starts with two subsets, each of which contains nodes from exactly one element of any matching pair. The aim is to interchange elements between the two until, according to some measure, both contains highly correlated elements. Let us denote this measure by  $\varpi_S$ , where  $S$  is a set of qubit indices, and define it as the sum of absolute values of all the Pearson correlation coefficients corresponding to (unordered) pairs of elements within subset  $S$ , i.e.

$$\varpi_S = \sum_{\substack{i,j \in S \\ i < j}} |q_{ij}|. \quad (51)$$

Now, choose randomly a subset  $S_L$  in such a way that it contain exactly one element from each pair of the matching. Of course, the complementary subset  $S_R = \{1, 2, \dots, n\} - S_L$  contain at most one element of each pair (always the different one that  $S_L$  does), but it also contains elements not picked up for matching pairs. Nevertheless, there should exist exactly one such element in  $S_R$ , if  $n$  is an odd number, and no such element should exist, if  $n$  is even.

Interchanging elements between  $S_L$  and  $S_R$  should be performed in such a way that the overall measure

$$\varpi = \varpi_{S_L} + \varpi_{S_R} \quad (52)$$

increases. Denoting the subsets of qubit indices at time  $t$  by, respectively,  $S_L^t$  and  $S_R^t$ , and choosing a pair  $l \in S_L^t$  and  $r \in S_R^t$ , where  $\{l, r\}$  is a pair from the matching, to interchange between  $S_L^t$  and  $S_R^t$ , we have:

$$\begin{aligned} S_L^{t+1} &= S_L^t \setminus \{l\} \cup \{r\}, \\ S_R^{t+1} &= S_R^t \setminus \{r\} \cup \{l\}. \end{aligned} \quad (53)$$

We also expect that

$$\varpi^{t+1} = \varpi_{S_L^{t+1}} + \varpi_{S_R^{t+1}} \geq \varpi^t = \varpi_{S_L^t} + \varpi_{S_R^t}, \quad (54)$$

that is, the interchange must improve the measure  $\varpi$  of “internal correlations” between times  $t$  and  $t + 1$ . With  $q = [q_{ij}]_{i,j \in \{1, \dots, n\}}$  as per (47), the improvement by interchanging  $l$  with  $r$  is  $\Delta\varpi(l, r) = \varpi^{t+1} - \varpi^t =$

$$= \sum_{i \in S_L^t \setminus \{l\}} (|q_{ir}| - |q_{il}|) + \sum_{i \in S_R^t \setminus \{r\}} (|q_{il}| - |q_{ir}|) =$$

$$= \sum_{i \in \{1, 2, \dots, n\} \setminus \{l, r\}} (-1)^{[i \in S_R^t]} (|q_{ir}| - |q_{il}|). \quad (55)$$

In each step, this greedy algorithm looks for a pair  $(l, r)$  such that  $\Delta\varpi(l, r)$  is greatest, yet positive. It then interchanges  $l$  and  $r$  between  $S_L$  and  $S_R$ . It does so unless the greatest  $\Delta\varpi(l, r)$  is zero or negative, which shows no possible improvement in one step by interchanging just one pair of qubit indices.

Of course, one could argue that this way some local optimum can be hit, while the global one is missed. This might be true, but a full scan would entail  $2^{\lfloor \frac{n}{2} \rfloor}$  computations similar to (55). Each of the  $2^{\lfloor \frac{n}{2} \rfloor}$  possibilities means that either endpoint of each of  $\lfloor \frac{n}{2} \rfloor$  matching pairs is selected for  $S_L$ . Thus, the complexity grows to at least exponential in size.

#### Example 4

Let  $n, q$  and the corresponding matching be as in example 3. Assume that, according to the matching, the two initial subsets are  $\{1, 2, 5\}$  and  $\{3, 4, 6\}$  – see figure 4.

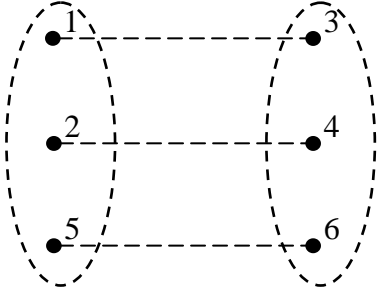


Fig. 4. Initial split of qubits

Now, the potential impact of switching of the three pairs, (1,3), (2,4) and (5,6), should be assessed. That is, there are 3 following cases of potential splits to investigate: a) {2,3,5} and {1,4,6} after interchanging 1 and 3 between  $S_L^0 = \{1,2,5\}$  and  $S_R^0 = \{3,4,6\}$ , b) {1,4,5} and {2,3,6} after interchanging 2 and 4, and c) {1,2,6} and {3,4,5} after interchanging 5 and 6. Hence, considered should be three values:  $\Delta\varpi(1,3)$ ,  $\Delta\varpi(2,4)$  and  $\Delta\varpi(5,6)$ . These are, respectively,

$$\begin{aligned} \Delta\varpi(1,3) &= 0 - 1 + \frac{\sqrt{2}}{\sqrt{3}} - 0 + 0 - 1 + 0 - \frac{1}{3} = \\ &= \frac{\sqrt{6}-7}{3} \cong -1.5117, \end{aligned}$$

$$\begin{aligned} \Delta\varpi(2,4) &= 0 - 1 + \frac{\sqrt{2}}{\sqrt{3}} - 0 + 0 - 1 + 0 - \frac{1}{3} = \\ &= \frac{\sqrt{6}-7}{3} \cong -1.5117, \end{aligned}$$

$$\begin{aligned} \Delta\varpi(5,6) &= 0 - 0 + 0 - 0 + \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{3} + \frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{3} = \\ &= \frac{2\sqrt{6}-2}{3} \cong 0.966. \end{aligned}$$

This shows that,  $\Delta\varpi(5,6)$  is the greatest one and is positive. Hence, 5 and 6 should be interchanged with each other between the two subsets, which now are going to be {1,2,6} and {3,4,5} and the matching – as follows:

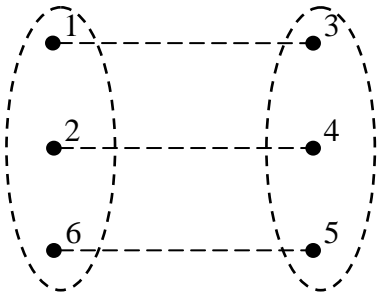


Fig. 5. The split after the first turn

In the next turn, computed are: a)  $\Delta\varpi(1,3)$  to assess the potential splitting resulting in {2,3,6} with {1,4,5}, b)  $\Delta\varpi(2,4)$  to assess potential {1,4,6} with {2,3,5}, and c)  $\Delta\varpi(6,5)$  to assess {1,2,5} with {3,4,6}. The last case, of course, would be a step back, so we do not expect a positive value of  $\Delta\varpi(6,5)$  and would even omit computations in this case. In current

turn  $\Delta\varpi(1,3) \cong -2.483$ ,  $\Delta\varpi(2,4) \cong -2.483$ , and  $\Delta\varpi(6,5) \cong -0.966$ . Now, for all pairs  $(l,r)$ ,  $\Delta\varpi(l,r) < 0$  and no improvement can be achieved further by a single interchange. ▲

## 6. Unbalanced splitting algorithm

While the balanced splitting algorithm guarantees that a reasonable split is made, it does not guarantee that the split is optimal. That is, splitting a quantum register into two, almost equal in the number of qubits, allows to save the most memory although, at the same time, a lot of information on entanglement (or correlations) might be lost. However, another split might perform better in terms of the information lost, yet still saving some memory storing the information on the entanglement (or correlations). So, splits other than balanced ones should be considered.

Recall that the Pearson correlation matrix shows the strength of the entanglement (or correlation) between each pair of qubits in a register. Unbalanced splitting algorithm starts with singletons of each qubit, joining them next step-by-step into groups, until there are finally only 2 groups left. First, the Pearson correlation coefficients are sorted in a descending order, then each iteration over these coefficients determines which two current sets should be joined together to form a bigger one. Of course, the iteration breaks as soon as the number of the sets falls to 2. In essence, two different sets, each of which contains one of the two elements from the (unordered) pair corresponding to the current Pearson correlation coefficient, are joined. This is shown in the following example:

### Example 5

Let  $n, \varrho$  be as in example 3. The Pearson correlation coefficients descending order determine the following order of pairs of qubit indices: (3,4), (1,2), (4,5), (3,5), (5,6), (4,6), (3,6), (2,6), (2,5), (2,4), (2,3), (1,6), (1,5), (1,4), (1,3). The initial singleton sets are formed from all the elements (qubits):

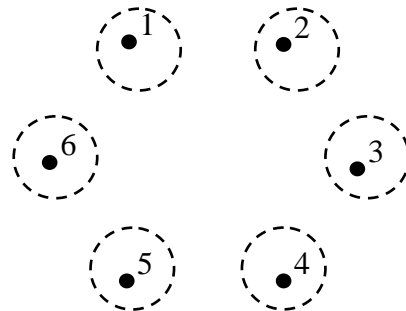


Fig. 6. Initial singleton sets for unbalanced splitting

The first pair (3,4) makes the singletons {3} and {4} be joined together:

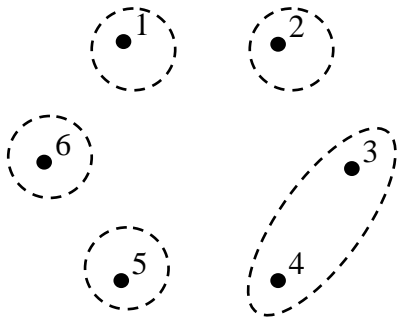


Fig. 7. Step 1 of the unbalanced splitting

Next, pair (1,2) gives:

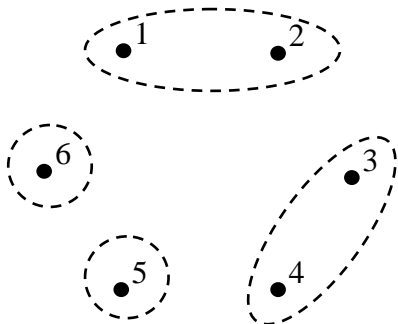


Fig. 8. Step 2 of the unbalanced splitting.

At step 3, pair (4,5) joins together the set {3,4} containing 4 and the set {5} containing 5:

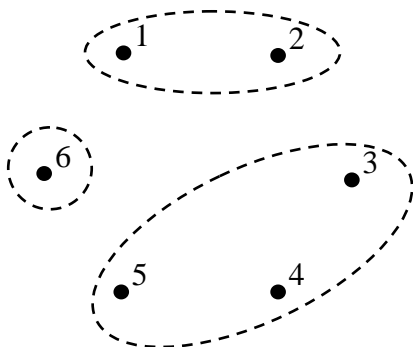


Fig. 9. Step 3 of the unbalanced splitting

Pair (3,5) does not join any two different sets, as its both elements belong to an already build subset {3,4,5}. Finally, pair (3,6) makes subsets {6} and {3,4,5} be joined into {3,4,5,6}.

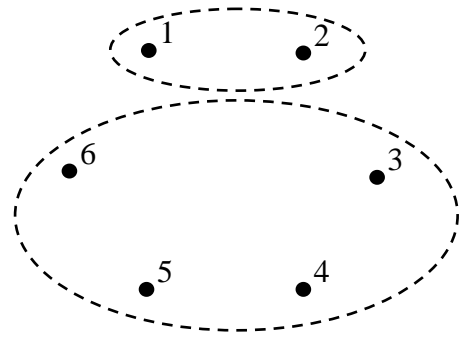


Fig. 10. The last step of the unbalanced splitting

This, breaks the iteration over the pairs, as the current splitting has the desired number of subsets – two. These are {1,2} and {3,4,5,6}. ▲

**Example 6**

Let a separable quantum state be  $|\varphi\rangle =$   
 $\left(\frac{1}{2}|0000\rangle + \frac{1}{2}|1000\rangle + \frac{1}{2}|1100\rangle + \frac{1}{2}|1111\rangle\right)$   
 $\otimes \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) =$   
 $= \frac{1}{2\sqrt{2}}|000000\rangle + \frac{1}{2\sqrt{2}}|100000\rangle +$   
 $+ \frac{1}{2\sqrt{2}}|110000\rangle + \frac{1}{2\sqrt{2}}|111100\rangle +$   
 $+ \frac{1}{2\sqrt{2}}|000011\rangle + \frac{1}{2\sqrt{2}}|100011\rangle +$   
 $+ \frac{1}{2\sqrt{2}}|110011\rangle + \frac{1}{2\sqrt{2}}|111111\rangle.$

Measuring the first two qubits, it is easily seen that they collapse to  $|0\rangle, |0\rangle$  or  $|1\rangle, |1\rangle$  with probabilities  $p_{12}^{00} = p_{12}^{11} = 0.5$ . Of course,  $p_{12}^{01} = p_{12}^{10} = 0$ . The formula (44) gives  $q_{12} = 1$ . Measuring qubits no. 1 and 3, the following probabilities are found:  $p_{13}^{00} = p_{13}^{10} = \frac{3}{8}$  and  $p_{13}^{01} = p_{13}^{11} = \frac{1}{8}$ , giving  $q_{13} = 0$ , etc. The following table shows all the results:

Tab. 2. Example probabilities and correlation coefficients

$i$	$j$	$p_{ij}^{00}$	$p_{ij}^{01}$	$p_{ij}^{10}$	$p_{ij}^{11}$	$q_{ij}$
1	2	0.5	0	0	0.5	1
1	3	0.375	0.125	0.375	0.125	0
1	4	0.375	0.125	0.375	0.125	0
1	5	0.25	0.25	0.25	0.25	0
1	6	0.25	0.25	0.25	0.25	0
2	3	0.375	0.125	0.375	0.125	0
2	4	0.375	0.125	0.375	0.125	0
2	5	0.25	0.25	0.25	0.25	0
2	6	0.25	0.25	0.25	0.25	0
3	4	0.75	0	0	0.25	1
3	5	0.5	0.25	0	0.25	$\sqrt{2/3}$
3	6	0.25	0.5	0	0.25	$1/3$
4	5	0.5	0.25	0	0.25	$\sqrt{2/3}$
4	6	0.25	0.5	0	0.25	$1/3$
5	6	0.25	0.25	0	0.5	$\sqrt{1/3}$

It is now clear that the correlation matrix  $q = [q_{ij}]_{i,j \in \{1,\dots,6\}}$  in example 3 resulted from the (separable) quantum state shown here. Example 5, in turn, shows that unbalanced splitting algorithm splits the quantum register exactly as per separability of this state (qubits no. 1 and 2 vs. qubits no. 3, 4, 5, and 6).  $\blacktriangle$

## 7. Computational complexity

Both balanced and unbalanced splitting algorithms start with computing Pearson correlation coefficients (or a matrix of them). Since  $q_{ii} = 1$  and  $q_{ij} = q_{ji}$ , effectively, there are  $\frac{n(n-1)}{2} = \mathcal{O}(n^2)$  of them to compute. Each one needs a constant number of operations, that is  $\mathcal{O}(1)$  to derive as per (44), but the four underlying values  $p_{ij}^{00}$ ,  $p_{ij}^{01}$ ,  $p_{ij}^{10}$ , and  $p_{ij}^{11}$ , need to iterate together through all the probability amplitudes, as per (32). Note that each one of the above four needs one fourth of the amplitudes. This gives  $2^n$  operations. Overall time complexity for computing the correlations is

$$\mathcal{O}(n^2) \cdot \mathcal{O}(1) \cdot \mathcal{O}(2^n) = \mathcal{O}(n^2 2^n). \quad (56)$$

Both algorithms need also sorting the coefficients, whether in ascending or descending order. Currently, the best known universal sorting algorithms' complexity is  $\mathcal{O}(x \log x)$  where  $x$  is the problem size. Since there are  $\mathcal{O}(n^2)$  Pearson correlation coefficients to sort, this gives the complexity of

$$\mathcal{O}(n^2 \log n^2) = \mathcal{O}(n^2 \log n). \quad (57)$$

Thus the overall complexity of preparing the Pearson correlation coefficients for further usage in either of the algorithms is

$$\mathcal{O}(n^2 2^n) + \mathcal{O}(n^2 \log n) = \mathcal{O}(n^2 2^n). \quad (58)$$

Now this is where comes the difference between the two algorithms. To find a matching, a loop over all  $\mathcal{O}(n^2)$  coefficients is necessary. Assuming that each node is flagged as soon as it is picked up for a next pair in the constructed matching, checking whether a particular node has already been picked up is straightforward and consumes only  $\mathcal{O}(1)$  of time. Thus the matching is constructed within

$$\mathcal{O}(n^2) \cdot \mathcal{O}(1) = \mathcal{O}(n^2) \quad (59)$$

operations and  $\lfloor \frac{n}{2} \rfloor$  pairs are produced.

It can be assumed that a canonical split is done as follows: the subset  $S_L$  takes the lower values from each matching pair and the subset  $S_R$  takes the greater ones. Any interchange can be expressed by setting a special flag on a pair that means “ $S_L$  takes the greater value from this pair, while  $S_R$  takes the lower one”. Thus a table of  $\lfloor \frac{n}{2} \rfloor = \mathcal{O}(n)$  flags would encode the resulting split over the previously prepared matching. In each iteration improving the measure (52), the formula (55) has to be applied to each of the  $\lfloor \frac{n}{2} \rfloor$  pairs. There are  $(n-2)$  operations involved in computing (55), assuming that the matrix  $q$  has been already known. Since finding a best value is linear with the number of items, it takes  $\lfloor \frac{n}{2} \rfloor (n-2) = \mathcal{O}(n^2)$  operations to find a next pair to interchange the values between  $S_L$  and  $S_R$ . The space of this optimisation problem consists of  $2^{\lfloor \frac{n}{2} \rfloor} = \mathcal{O}(2^n)$  possible solutions. No solutions can be “visited” twice, otherwise it would mean that no improvement is made between two consecutive steps. Thus the number of operations to find a balanced matching, given sorted coefficients  $q_{ij}$ , is limited to

$$\lfloor \frac{n}{2} \rfloor (n-2) 2^{\lfloor \frac{n}{2} \rfloor} = \mathcal{O}(n^2 2^n) \quad (60)$$

operations. In total, the complexity of the balanced algorithm is

$$\mathcal{O}(n^2 2^n) + \mathcal{O}(n^2 2^n) = \mathcal{O}(n^2 2^n). \quad (61)$$

The unbalanced algorithm, iterates over  $\frac{n(n-1)}{2}$  coefficients  $q_{ij}$ . Let us assume that its data structure to represent the subset a particular node belongs to is a table that maps each node number to an integer representing the subset number. This is a random access table, at no cost of reading and writing to it, that is  $\mathcal{O}(1)$ . In each iteration, two nodes,  $i$  and  $j$ , corresponding to a pair  $q_{ij}$ , determine if two subsets need to be joined. If so, an inner loop iterating through all the  $n = \mathcal{O}(n)$  nodes in the aforementioned map lets necessary remapping. The stop condition of the outer loop requires maintaining the current number of subsets. There are  $n$  of them when the outer loop starts, and decreases by 1, if a pair  $(i, j)$  determines joining two different subsets. Hence, this is straightforward to stop when the number of subsets falls to 2, with the cost of such a check limited to  $\mathcal{O}(1)$ . The cost of unbalanced splitting algorithm, given sorted coefficients  $q_{ij}$ , is thus

$$\mathcal{O}\left(\frac{n(n-1)}{2}\right) \cdot \mathcal{O}(n) = \mathcal{O}(n^3). \quad (62)$$

However, in total, it takes

$$\mathcal{O}(n^2 2^n) + \mathcal{O}(n^3) = \mathcal{O}(n^2 2^n). \quad (63)$$

This seems too much, but keeping register sizes within reasonable limits would allow to keep the complexity reasonable as well.

Having determined the potential split, the disentanglement measure should be computed so as to assess it. The formula (10) shows that there are  $2^d$  coefficients  $A_k$ , each computed as per (12) that involves  $2 \cdot 2^{n-d} + 1$  algebraic operations on  $\gamma_{kl}$ s. This gives

$$2^d \cdot (2 \cdot 2^{n-d} + 1) = \mathcal{O}(2^n) \quad (64)$$

operations. Similarly, (11) together with (13) shows that computing all  $B_l$ s need that same number of algebraic operations,  $\mathcal{O}(2^n)$ .

## 8. Conclusion

This paper has shown that a quantum register state can be approximated by some separable state, close enough to the original one in the space of quantum states. To help measure the proximity, the inner product of the two states is used. However, the model described here is applicable to quantum states of real nonnegative quantum probability amplitudes. Hence, further research can be conducted to discover similar

properties and rights in a wider domain of complex amplitudes.

The splitting routines, the balanced and unbalanced ones, might also be subject to further research, to find out if there are better ones. That is, it is desirable that the time complexity of a potential splitting routine is moderate, yet giving sufficient results. Special care should be taken to ensure that separable states are (in most cases) correctly recognized and a quantum register is split as per its separable form of the original state (expressed as a tensor product).

The process of splitting a register should not lead to a completely disentangled register, i.e. a one, in which every qubit form a different group, each containing only one element. A reasonable limit on such a process should be imposed, based on the disentanglement measure.

The relatively high cost of splitting algorithms should be eased by keeping the separable subregisters small enough. It means that, even if the number of qubits in a register is relatively high, these qubits should be kept in groups small “enough” to split them and join as needed. The “mix” of splitting and joining the groups of qubits, as a dynamically reconfigurable quantum state model, is beyond the scope of this paper, though.

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## **Modelowanie rozplątania rejestru kwantowego**

P.A. RYSZAWA

Implementacja kwantowo-inspirowanych algorytmów na komputerach klasycznych musi godzić sprzeczność pomiędzy koniecznością oszczędzania pamięci operacyjnej a ilością pamięci potrzebnej na reprezentację stanu kwantowego z potencjalnym splątaniem. Wiadomo, że to ostatnie pochłania zasoby pamięciowe w ilości wykładniczej wraz ze wzrostem liczby kubitów. Niniejszy artykuł zarysowuje ideę istotnej redukcji potrzebnych zasobów pamięciowych, zniekształcającej przy tym reprezentację oryginalnego stanu tylko nieznacznie lub wcale. W chwili obecnej, rozważane są nieujemne rzeczywiste amplitudy prawdopodobieństwa.

**Słowa kluczowe:** obliczenia kwantowe, splątanie kwantowe, rozplątanie, rejestr kwantowy.